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General common fixed point theorems in fuzzy metric spaces

K.P.R.Rao
Department of Applied Mathematics
Acharya Nagarjuna University - Dr. M.R.Appa Row Campus
Nuzvid-521 201,Krishna Dt.,A.P.,INDIA.
Email: kprrao2004@yahoo.com

K.V.Siva Parvathi
Department of Applied Mathematics
Acharya Nagarjuna University - Dr. M.R.Appa Row Campus
Nuzvid-521 201,Krishna Dt.,A.P.,INDIA.
Email: kvsp1979@yahoo.com

Abstract. In this paper , we obtain two general common fixed point theorems for two maps in fuzzy metric spaces.

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1. Introduction and preliminaries

The theory of fuzzy sets was introduced by L.Zadeh [9] in 1965.George and Veeramani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [7].Grabiec[10] proved the contraction principle in the setting of fuzzy metric spaces introduced in [1].For fixed point theorems in fuzzy metric spaces some of the interesting references are[1,3,4,5,10,12-17,19,20]. In the sequel, we need the following

Definition 1. [2] A binary operation $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is a continuous t-norm if it satisfies the following conditions

- **(1):** $*$ is associative and commutative,
- **(2):** $*$ is continuous,
- **(3):** $a*1=a$ for all $a\in[0,1]$,
- **(4):** $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for each $a,b,c,d\in[0,1]$.

Two typical examples of continuous t-norm are $a*b=ab$ and $a*b=min\{a,b\}$.

Definition 2. [1]A 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$, satisfying the following conditions for each $x,y,z\in X$ and each t and $s>0$,

- **(1):** $M(x,y,t)>0$,
- **(2):** $M(x,y,t)=1$ if and only if $x=y$,
- **(3):** $M(x,y,t)=M(y,x,t)$,
- **(4):** $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,

(5): $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 3. [10] *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Definition 4. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, i.e., whenever

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 5. [8]*Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0, \infty)$.*

Definition 6. [1]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$. The sequence $\{x_n\}$ is said to be Cauchy if $\lim_{n,m\to\infty} M(x_n, x_m, t) = 1$. The space $(X, M, *)$ is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 7. [18]. A fuzzy metric space $(X, M, *)$ is called precompact if for each $0 < r < 1$ and each $t > 0$, there is a finite subset $A \in X$ such that $X = \bigcup_{a\in A} B(a, r, t)$. A fuzzy metric space $(X, M, *)$ is called compact if (X, τ) is a compact topological space. It is clear that every compact set is closed and F-bounded.

Definition 8. [6].Let f and g be self mappings on a fuzzy metric space (X, d). Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is,$fx = gx$ implies that $fgx = gfx$.

Generally, several authors obtained fixed point theorems in fuzzy metric spaces for a single map using one of the following contraction conditions.
There exists $k \in (0, 1)$ such that for all $x, y \in X$ and for all $t > 0$,
(1)$M(Tx, Ty, kt) \geq M(x, y, t)$,
(2)$M(Tx, Ty, kt) \geq min \left\{ \begin{array}{c} M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ M(x, Ty, 2t), M(y, Tx, t) \end{array} \right\}$,
(3)$M(Tx, Ty, kt) \geq min \left\{ \begin{array}{c} M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ M(x, Ty, 2t), M(y, Tx, 2t) \end{array} \right\}$,
(4)$M(Tx, Ty, kt) \geq min \left\{ \begin{array}{c} M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ M(x, Ty, \alpha t), M(y, Tx, (2-\alpha)t) \end{array} \right\}, \forall\, \alpha \in (0, 2)$.
In all these types of theorems ,the authors assumed that $\lim_{t\to\infty} M(x, y, t) = 1$, $\forall x, y \in X$.

In this paper, without using this condition, we prove the following general common fixed point theorem in F-bounded fuzzy metric spaces.

2. MAIN RESULTS

Theorem 9. *Let T and f be self maps of on a F-bounded fuzzy metric space $(X, M, *)$ satisfying*
(9.1) $T(X) \subseteq f(X)$, (T, f) *is a weakly compatible pair and $f(X)$ is complete,*

$$(9.2)\ M(Tx, Ty, t) \geq \phi\left(min\left\{\begin{array}{c} M(fx, fy, t), M(fx, Tx, t), M(fy, Ty, t), \\ M(fx, Ty, t), M(fy, Tx, t) \end{array}\right\}\right)$$

for all $x, y \in X$ and for all $t > 0$, where $\phi : [0, 1] \to [0, 1]$ is continuous and monotonically increasing such that $\phi(s) > s$, for all $s \in [0, 1)$.
Then f and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From (9.1), there exists a sequence $\{x_n\}$ in X such that
$Tx_n = fx_{n+1} = y_n$, say.
Case(i): Suppose $y_{n+1} = y_n$ for some n.
Then $Tz = fz$, where $z = x_{n+1}$. Denote $p = Tz = fz$.
Since (T, f) is a weakly compatible pair,we have $Tp = fp$.
From (9.2), we have

$$\begin{aligned} M(Tp, p, t) &= M(Tp, Tz, t) \\ &\geq \phi\left(\min\left\{\begin{array}{c} M(fp, fz, t), M(fp, Tp, t), M(fz, Tz, t), \\ M(fp, Tz, t), M(fz, Tp, t) \end{array}\right\}\right) \\ &= \phi\left(\min\left\{M(Tp, p, t), 1, 1, M(Tp, p, t), M(Tp, p, t)\right\}\right) \\ &= \phi\left(M(Tp, p, t)\right) \\ &> M(Tp, p, t)),\ if\ M(Tp, p, t) < 1. \end{aligned}$$

Hence $Tp = p$. Thus $fp = Tp = p$.
If q is another common fixed point of f and T, then

$$\begin{aligned} M(p, q, t) &= M(Tp, Tq, t) \\ &= \phi\left(\min\left\{M(p, q, t), 1, 1, M(p, q, t), M(p, q, t)\right\}\right) \\ &= \phi\left(M(p, q, t)\right) \\ &> M(p, q, t)\ if\ M(p, q, t) < 1 \end{aligned}$$

Hence $p = q$. Thus p is the unique common fixed point of f and T.
Case(ii): Assume that $y_{n+1} \neq y_n$ for all $n \in \mathbb{N}$.
For $n \in \mathbb{N}$, let $\alpha_n(t) = inf\{M(y_i, y_j, t) : i \geq n, j \geq n\}$ for all $t > 0$.
Then $\{\alpha_n(t)\}$ is a monotonically increasing sequence of real numbers between 0 and 1 for all $t > 0$.
Hence $\lim\limits_{n\to\infty} \alpha_n(t) = \alpha(t)$ for some $0 \leq \alpha(t) \leq 1$.
For any $n \in \mathbb{N}$ and integers $i \geq n$, $j \geq n$, we have

$$\begin{aligned} M(y_i, y_j, t) &= M(Tx_i, Tx_j, t) \\ &\geq \phi\left(\min\left\{\begin{array}{c} M(y_{i-1}, y_{j-1}, t), M(y_{i-1}, y_i, t), M(y_{j-1}, y_j, t), \\ M(y_{i-1}, y_j, t), M(y_{j-1}, y_i, t) \end{array}\right\}\right) \\ &\geq \phi\left(\alpha_{n-1}(t)\right),\ since\ \phi\ is\ monotonically\ increasing \end{aligned}$$

Hence $\alpha_n(t) \geq \phi(\alpha_{n-1}(t))$.
Letting $n \to \infty$, we get $\alpha(t) \geq \phi(\alpha(t)) > \alpha(t)$, if $\alpha(t) < 1$.
Hence $\alpha(t) = 1$ so that $\lim\limits_{n\to\infty} \alpha_n(t) = 1$.
Thus for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\alpha_n(t) > 1 - \epsilon$ for all $n \geq n_0$.
Thus for $n \geq n_0$, $m \in \mathbb{N}$, we have $M(y_n, y_{n+m}, t) > 1 - \epsilon$.

Hence $\{y_n\}$is a Cauchy sequence in X.Since $f(X)$ is complete, it follows that $y_n \to z$ for some $z \in f(X)$.Hence there exists $u \in X$ such that $z = fu$. Now,

$$M(Tu, Tx_n, t) \geq \phi \left(\min \left\{ \begin{array}{c} M(fu, fx_n, t), M(fu, Tu, t), M(fx_n, Tx_n, t), \\ M(fu, Tx_n, t), M(fx_n, Tu, t) \end{array} \right\} \right)$$

Letting $n \to \infty$, we get

$$\begin{aligned} M(Tu, z, t) &\geq \phi \left(\min \{1, M(z, Tu, t), 1, 1, M(z, Tu, t)\} \right) \\ &= \phi(M(z, Tu, t)) > M(z, Tu, t) \; if \; M(z, Tu, t) < 1 \end{aligned}$$

Hence $Tu = z$. Thus $fu = Tu = z$.

The rest of the proof follows as in case(i). □

Corollary 10. *Let T be self map of on a F-bounded complete fuzzy metric space $(X, M, *)$ satisfying*

$$M(Tx, Ty, t) \geq \phi \left(min \left\{ \begin{array}{c} M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ M(x, Ty, t), M(y, Tx, t) \end{array} \right\} \right)$$

for all $x, y \in X$ and for all $t > 0$, where $\phi : [0, 1] \to [0, 1]$ is continuous and monotonically increasing such that $\phi(s) > s$, for all $s \in [0, 1)$.
Then T has a unique fixed point in X.

Now using the technique of Shih and Yeh [11] in metric spaces, we prove the following theorem in compact fuzzy metric spaces.

Theorem 11. *Let $(X, M, *)$ be a compact fuzzy metric space , $f, T : X \to X$ be satisfying*
$(11.1)T$ is continuous, $fT = Tf$ and
(11.2) $M(Tx, Ty, t) > \min \{M(x_1, y_1, t) : x_1, y_1 \in O(x) \cup O(y)\}$
for all $x, y \in X$ with $x \neq y, \forall t > 0$, where
$O(x) = \{hx : h \in \tau\}, \tau$ being the semi group of self maps on X generated by $\{f, T, I\}$ (I is the identity map on X).
Then f and T have a unique common fixed point $z \in X$.

Proof. We know that T^nX is compact and $T^{n+1}X \subseteq T^nX$ for $n = 1, 2, 3, ...$
Let $X_0 = \bigcap\limits_{n=1}^{\infty} T^nX$.
Then X_0 is a nonempty compact subset of X,$TX_0 = X_0$ and $fX_0 \subseteq X_0$.
Since M is continuous on $X_0^2 \times (0, \infty)$ and X_0 is compact, it follows that for each $t > 0, M(., ., t)$ has a minimum value. Hence there exist $z_1, z_2 \in X_0$ such that $M(z_1, z_2, t) = inf\{M(x, y, t) : x, y \in X_0\}$ for each $t > 0$.
Since $TX_0 = X_0$, there exist $x_1, x_2 \in X_0$ such that $Tx_1 = z_1$ and $Tx_2 = z_2$.
Suppose $x_1 \neq x_2$. Then from (11.2),we have

$$\begin{aligned} M(z_1, z_2, t) &= M(Tx_1, Tx_2, t) \\ &> \min \{M(x, y, t) : x, y \in O(x_1) \cup O(x_2)\} \\ &\geq M(z_1, z_2, t) \end{aligned}$$

It is a contradiction.
Hence $x_1 = x_2$ and so $z_1 = z_2$. Hence X_0 is a singleton set, say, $\{z\}$.
Thus z is a common fixed point of f and T.
From (11.2), it is clear that z is the unique common fixed of f and T. □

Corollary 12. *Let T be a continuous self map on a compact fuzzy metric space $(X, M, *)$ satisfying*

$$M(Tx, Ty, t) > min \left\{ \begin{array}{c} M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ M(x, Ty, t), M(y, Tx, t) \end{array} \right\}$$

for all $x, y \in X$ *with* $x \neq y$ *and for all* $t > 0$.
Then T *has a unique fixed point in* X.

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